Common Errors in Mental Computation of Students in Grades 3 – 10

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Error patterns in computation have been well researched for written computation; but there is little research into the error patterns common among students computing mentally. The responses made by 3035 students in Grades 3 to 10 were analysed and the most common errors made at each grade level ascertained. Common patterns of errors are described. Possible reasons for these errors and implications for the classroom are discussed.

Mental computation is officially recognised as requiring at least equal attention in school mathematics with written computation. For example the *National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991) states:

> People need to carry out straightforward calculations mentally, and students should regard mental arithmetic as a first resort in many situations where a calculation is needed. Strategies associated with mental computation should be developed explicitly throughout the schooling years and should not be restricted to the recall of basic facts...less emphasis should be given to standard paper-and-pencil algorithms and, to the extent that they continue to be taught, they should be taught at later stages in schooling. (p. 109)

A number of major attempts to develop coherent approaches to the teaching of mental computation are under way both in Australia and abroad. In the United Kingdom, the National Numeracy Policy (available on the website: http://www.standards.dfee.gov.uk/) states that "An ability to calculate mentally lies at the heart of numeracy". In the Netherlands a systematic attempt is being made to develop and link mental and informal written computation methods (see for example, Beishuizen, 1993). However much research is needed to secure these approaches on a sound footing. As one example, there is nothing in the mental computation literature that parallels Ashlock's *Error Patterns in Computation* (1994), which confines itself exclusively to errors in written computation. "This entire book is designed to help you learn as much as possible from the written work of children" (Ashlock, 1994, p. 13). Since children's focus of thought, and consequently their patterns of thinking, are often markedly different when they are engaged in mental computation from those they employ when calculating with pencil-and-paper, it is to be expected that the kinds of errors they make, and the reasons for these errors, may also sometimes differ.

Bana, Farrell, and McIntosh (1995) and Bana, Farrell, and McIntosh (1997) describe errors made in mental computation of whole numbers, fractions, decimals and percents by students in years 3, 5, 7, and 9, but there appear to have been no follow-up studies. Moreover the sample sizes were quite small (approximately 160 at each age level) and the number of questions very restricted (30 at each of grades 3 and 5 and 40 at grades 7 and 9).

B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 457-464). Sydney: MERGA.

The Present Study

The current study used a considerably greater number of questions and a much larger student sample than that used in the research cited above. A total of 3035 students in grades 3 to 10 answered questions from a pool of 244 mental computation items, covering whole numbers, fractions, decimals and percents. Four parallel tests were developed at each of four levels: grades 3/4, 5/6, 7/8 and 9/10. The test for grades 3/4 contained 50 items; tests for all other grades contained 65 questions. There were overlapping questions between each pair of the four parallel tests at each grade, and between grade levels.

The items were developed and extended from tests used in a previous study (Callingham & McIntosh, 2001). In both studies, all tests were recorded on audiotape to ensure consistency of timing for each item. All items were administered orally via the cassette recorder, and were not seen by the students. Answers only were written on the answer sheet with no written working allowed. Class teachers administered the tests.

In the previous study students had either five seconds or fifteen seconds to answer each item. This was intended to distinguish Short items (five seconds) for which the students had instant recall from Long items (fifteen seconds) which they could calculate, given time. However it was clear that many students could calculate the answer in five seconds. In the present study therefore the time for Short items was reduced to three seconds, following the procedure used in New Zealand assessments (Flockton & Crooks, 1997). In addition some five-second items were included in order to enable comparisons to be made with the previous study.

The items were restricted to the range of calculations that it was considered desirable for most students up to year 10 to be able to calculate mentally. There was hence no attempt to include technically difficult calculations that might only be within the capability of a few gifted students. Rather the items were designed to be capable of calculation by a student who had conceptual understanding of the numbers and operation involved. For example the 'hardest' addition of whole numbers was 79 + 26; the hardest fraction addition was 1/2 + 1/3; the hardest decimal item was 0.2 ÷ 5 (Grades 9/10 only); whereas the hardest percent item was 33 1/3% of 600. It was expected that the majority of errors made would reveal weaknesses in understanding rather than inability to hold the complexity of the calculation in memory.

For each item, at each grade level, all incorrect answers were recorded, and note taken of the most common errors. These were then analysed for clusterings of error types; the most common of these error patterns for each number type (whole numbers, fractions, decimals, percents) are now described.

Error Patterns

Whole Number Errors

By far the most common error with addition and subtraction of whole numbers was an answer that differed by one from the correct answer; for example, for the 15-second item 27 – 9, 51 out of 98 incorrect answers given by Grade 3/4 students, and 21 out of 65 incorrect answers given by Grade 5/6 students, were either 17 or 19.

Table 1 gives the number of times an error of one was made for basic addition/subtraction fact items and for addition/subtraction of larger numbers. The table excludes all cases where no answer was given to a calculation.

For larger whole number addition/ subtraction items, the second most common error was an answer that was incorrect by 10. This error persisted through to Grade 9/10 students: for example, for the 15-second item 58 + 34, 21 out of 63 incorrect answers given by Grade 7/8 students, and 10 out of 28 incorrect answers given by Grade 9/10 students, were either 82 or 102.

Table 1
Errors of 1 for All Addition/Subtraction Items by Grade and Type of Calculation.

	Basic Fact Items				Larger Numbers			
	Grade 3/4	Grade 5/6	Grade 7/8	Grade 9/10	Grade 3/4	Grade 5/6	Grade 7/8	Grade 9/10
(a) Errors of 1	394	23	14	-	367	224	74	51
(b) All errors	1082	87	53	-	1506	1784	724	361
(a) as % of (b)	36.4	26.4	26.4	-	24.4	12.6	10.2	14.1

The most common error for multiplication and division basic facts was an answer that was wrong by one multiple: for example, for the item 21 ÷ 3, 21 out of 77 incorrect answers given by Grade 3/4 students, 21 out of 49 incorrect answers given by Grade 5/6 students, 6 out of 21 incorrect answers given by Grade 7/8 students, and 4 out of 15 incorrect answers given by Grade 9/10 students, were either 6 or 8.

Table 2 shows the number of times for each grade a wrong answer to a basic fact multiplication or division item was incorrect by one multiple, and expresses this as a percent of all errors for those items.

Table 2
Errors Incorrect by One Multiple for All Basic Fact Multiplication/Division Items by Grade.

	Basic Fact Items			
	Grade 3/4	Grade 5/6	Grade 7/8	Grade 9/10
(a) Errors of one multiple	279	319	229	80
(b) All errors	1294	1454	780	263
(a) as % of (b)	21.6	21.9	29.4	30.4

For multiplication and division of larger numbers, errors frequently occurred when one or both numbers was a multiple of 10. For example, of 143 students across grades 5 – 10 making errors for the item 9 x 200, 40 students gave the value 180.

Fraction Errors

Table 3 shows a range of the fraction items from the tests. Columns three to six show the percent of students at each grade answering the item correctly. The second column indicates the number of seconds given for each item. The most common incorrect answer across the grades is also given. Of 323 errors across grades 5-10 for the Long item 1 – 1/3, 101 gave either ¼ or 3/4. Of 121 errors across grades 7-10 for the item 3 ÷ 1/2, 81 gave 1 1/2.

The item 1 – 1/3 was given both as a Short (3-second) and as a Long (15-second) item. Table 3 shows that the change in the amount of time given made little difference to the percent of students answering the item correctly. The most common incorrect answers for this item for both timings were ¼ and ¾. However for the Short, but not the Long, item, the incorrect answer ½ was also common.

Table 3
Percent of Students Answering Correctly Selected Fraction Items by Grade, and Most Common Incorrect Answers

Item	Time (seconds)	Grade 3/4	Grade 5/6	Grade 7/8	Grade 9/10	Most Common Incorrect Answers
1/2 + 1/4	3	26	53	67	67	2/6 or 1/3
1 – 1/3	3	-	33	44	56	1/4, 3/4 or 1/2
1 – 1/3	15	-	30	51	52	1/4 or 3/4
1/2 + 1/3	15	-	-	8	10	3/4
1/2 of 1/2	15	-	35	55	78	1
1/2 ÷1/4	15	-	-	42	47	1/4
3 ÷ 1/2	15	-	-	46	474	1 1/2

Decimal Errors

Table 4 shows a range of the decimal calculations asked, together with the percent of students at each grade answering correctly. There were no decimal items in the tests for Grades 3/4. The second column indicates the number of seconds given for each item. The most common incorrect answer across the grades is also given.

Table 4
Percent of Students Answering Correctly Selected Decimal Items by Grade, and Most Common Incorrect Answers.

Item	Time (seconds)	Grade 5/6	Grade 7/8	Grade 9/10	Most Common Incorrect Answers
0.6 x 10	3	23	41	61	0.6
0.5 + 0.75	15	11	30	54	0.8
0.19 + 0.1	15	-	34	40	0.2
4.5 - 3	15	47	58	75	4.2
3 x 0.6	15	-	32	39	0.18
0.3 + 0.7	15	42	52	64	0.1

For the Short item 0.6 x 10, the incorrect answer 0.6 was given by a total of 270 students, and accounted for over 50% of all errors made at each of Grades 5/6, 7/8, and 9/10. For the Long item 0.3 + 0.7, the answer 0.1 constituted 221 of 296 incorrect answers.

Percent Errors

Table 5
Percent of Students Answering Correctly Selected Percent Items by Grade, and Most Common Incorrect Answers.

Item	Time (seconds)	Grade 5/6	Grade 7/8	Grade 9/10	Most Common Incorrect Answers
100% of 36	3	54	74	84	36% and 0
10% of 45	15	12	25	44	5
75% of 200	15	19	43	57	175
90% of 40	15	-	24	30	30
33 1/3% of 600	15	-	8	19	several
30% of 80	15	-	8	21	25, 20, 30, 50

Table 5 shows a range of the percent calculations asked, together with the percent of students at each grade answering correctly. There were no percent items in the tests for Grades 3/4. The second column indicates the number of seconds given for each item. The most common incorrect answer across the grades is also given.

Of 308 errors across grades 5 - 10 for the item 75% of 200, 59 gave 175. For most percent items, no one wrong answer was very common: instead a very wide range of incorrect answers was given. For example, for the item 30% of 80, the following, given in order of frequency, constituted 121 of the 177 incorrect answers: 25, 20, 30, 50, 15, 37.5, 27, 22, 28, 23, 3, 26, 60.

Discussion

A number of researchers have classified types of computational errors made by students, for example Brueckner (1930), Roberts (1968), Backman (1978), Engelhardt (1977), Brown and VanLehn (1982), Resnick (1984), and Ashlock (1994). However, except in discussing basic fact errors, the focus has been almost exclusively on categorising errors associated with written algorithms, errors that are often quite irrelevant to the study of mental computation. For example, 16 of 17 error patterns for whole number calculations described by Ashlock involve misunderstandings of the formats and procedures associated with the formal written algorithm for the operation.

Consideration of the errors associated with the present study suggests that the fundamental distinction that needs to be made for errors in mental computation is that between conceptual and procedural errors. A conceptual error is one made because the student does not understand sufficiently the nature of the numbers or the operation involved. A procedural error is one in which the student, although having an overall strategic understanding of what to do, makes either a careless error or other error in carrying out the strategy. For example, $0.1 \times 0.1 = 0.1$ and $3 \div ½ = 1\ 1/2$ are likely to be conceptual errors whereas $58 + 34 = 82$ and $3 \times 5 = 18$ are likely to be examples of procedural errors. (It is necessary to qualify the certainty of the classification, since one is placing an interpretation only on the child's thinking, however convinced one is of the validity of one's interpretation.). While procedural errors are associated with both written and mental computation, the procedures themselves, and therefore the types of errors, are often quite different. As an example, for the item 74 - 30, a quite common error at Grades 3/4, 5/6, and 7/8 in the current study was the answer 36. It is likely that students making this error had a correct overall procedure or strategy of taking the 4 off the 74, subtracting 30 from 70, and then replacing the 4; but a lack of control over the procedure led them to subtract rather than add the 4. It is interesting that this error did not occur with Grade 9/10 students, suggesting that this is a transitional error.

As observed in this study, the errors made by students with whole number calculations tended to be procedural, whereas those involving fractions, decimals, and percents were predominantly conceptual. For example, items such as 1 – 1/3, 0.3 + 0.7, and 30% of 80, which are typical of three types of items set in these categories, were very frequently answered incorrectly, and yet each depends on very simple arithmetical ability coupled with conceptual understanding of the type of number involved.

This study supports the finding of Kamii and Dominick (1997), that when children are performing calculations on whole numbers using their own, usually mental, strategies, their incorrect answers are usually reasonable (i.e., near the correct answer) in contrast to the answers given by students using written algorithms, which are often quite unreasonable.

Where an addition or subtraction was incorrect by one (and it is worth noting in passing that there were also frequent cases with larger numbers where the answer was wrong by two), it is difficult to avoid the conclusion that in many cases the children's strategy was to count up or down by ones; this is reinforced by the fact that this error occurred more often in basic fact calculations when the addend was larger. A recent study by the author has suggested that this error, after decreasing gradually in extent through Grades K to 3, appears to remain static at about 20% of errors in basic fact additions and subtractions in Grades 4 – 6.

A similar reason can be hypothesised for the number of multiplication/division errors that are wrong by one multiple. In both cases there appears to be an error of counting, whether by ones or by multiples of 2 to 10. However it is not clear how the failure to keep a correct tally of the count occurs. The author has been told anecdotally that in at least one Australian state skip counting traditionally starts with zero (e.g., 0, 3, 6, 9 …) and this can lead to an incorrect count if zero is then counted by the child as the first multiple. It would appear reasonable to suggest that the calculations that are incorrect by 10 are also often the result of a failure to keep an accurate record of counting in tens.

The most frequent errors associated with calculations involving fractions, decimals and percents appear to have a mainly conceptual basis.

The errors made in fraction computations scarcely overlap with those described by Ashlock (1994), who describes errors found with written computation, and ones in which the students could see the numbers written down, leading to more bizarre and intricate 'malgorithms' (Ruthven & Chaplin, 1998). The only exception is Ashlock's Error Pattern A-F-1 (p. 114), which is adding both numerators and denominators (e.g., 1/2 + ¼ = 2/6), and which was found frequently in the present study.

Not surprisingly, errors in mental computation of fractions appear to be much less intricate and, where their reasoning can be surmised, more conceptual. Three of the common errors given in Table 3 can be attributed to confusion of operations. However 1 – 1/3 = 1/4 or 3/4, and 1/2 + 1/3 = 3/4 appear linked to a tendency noticed by the author for children to intermingle and confuse thirds with quarters (1 – 1/3 = 1 – 1/4, or 1 – 3/4; 1/2 +

1/3 = 1/2 + 1/2): the result perhaps of hoping that thirds will be linked with halves and quarters with which children are more familiar.

Decimal computation errors appear to be predominantly associated with the common misunderstanding noted by Hart (1981) and Stacey and Steinle (1998), namely "thinking that the figures after the [decimal] point represented a 'different' number which also had tens, units etc" (Hart, 1981, pp. 51-52). Table 4 includes several examples of this error, including 0.5 + 0.75 = 0.8, and 3 x 0.6 = 0.18.

Errors with percents are less commonly analysed in the literature. Hart (1981) devotes only one page (p. 96) to percents, while Ashlock (1994) describes only two error patterns involving calculations of percents. Two observations may be made about the results given in Table 5: first, that the common errors are, for the most part, not significantly different from the correct answers, suggesting at least some basic number sense is present; second, that many students would appear not to move easily between percents and their fraction equivalents (75% = ¾, 30% = 3/10) as one way of simplifying calculations. Even at Grades 9/10, only 44% correctly calculated 20% of 15.

Implications for Teaching of Mental Computation

Major implications for teaching mental computation of whole numbers appear to concern development of efficient strategies, whereas for fractions, decimals, and percents the issue appears to be that of developing conceptual understanding.

For whole numbers, two issues should be addressed. First, when children are using counting strategies for computations, their teachers need to observe how they count and keep track of their counting, and need constantly to ask children how they arrived at their answers. Errors may be caused by inefficient use of their fingers or other procedural error, for example when adding 6 and 3, counting 6, 7, 8. If these incorrect procedures are not discovered and discussed early they can come ingrained and persist throughout primary school. Second, children need to be weaned off the increasingly inefficient strategy of counting on and back by ones, to more sophisticated, neater and simpler strategies: using doubles and near doubles, bridging ten, adding tens, using compatible numbers, using related known facts.

For fractions, decimals, and percents, the main remedy continues to lie in at least delaying the algorithmic teaching of procedures until children have a conceptual understanding of the objects and their operations. It appears that many children simply do not understand fraction, decimal and percent notation. For fractions, the work of Streefland (1991) is helpful. The website developed by Stacey and others at the University of Melbourne (http://online.edfac.unimelb.edu.au/485129/DecProj/index.htm) is an excellent resource for decimals. For percents, the approach developed by Dole (1999) provides a well-researched basis. One of the most urgent needs in mathematics education in Australia is to find effective ways to ensure that well developed approaches to number find their way particularly into the majority of middle school classrooms.

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